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Derivation of Collins' Formulas for Beam-Shape Distortion due to Sextupoles Using Hamiltonian Method*

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The introduction of sextupoles into a storage ring will distort the beam shape in both the horizontal and vertical phase space. The purpose of this note is to rederive the formulas for the lowest-order beam-shape distortion given by Collins[1] using the Hamiltonian approach such as the one used by Ohnuma[2]. Collins' formulas for the second-order tune-shift have been rederived by the Hamiltonian method in a recent note[3].

We shall go over the Hamiltonian method briefly for two reasons: (1) to make this note more readable and (2) to conform with the convention of Collins so that a comparison can be made.

We start from the Hamiltonian describing the motion of a single beam particle,

$$H_1 = \frac{1}{2}\left[P_x^2 + K_x(s)X^2\right] + \frac{1}{2}\left[P_y^2 + K_y(s)Y^2\right] + \frac{B_y''}{6B\rho}\left(X^3 - 3XY^2\right), \tag{1}$$

where X and Y denote the horizontal and vertical displacements from the ideal closed orbit at a distance s measured along the storage ring from some reference point, $P_x = dX/ds$ and $P_y = dY/ds$ are the corresponding canonical momenta, $K_x(s)$ and $K_y(s)$ are proportional to the restoring forces due to the ring's curvature and quadrupoles. The last term gives the normal-sextupole

*This note is written at the request of S. Ohnuma potential with $B\rho$ denoting the magnetic rigidity of the particle.

We next perform a canonical transformation into the Floquet space using the generating function

$$G_1(x, P_x, y, P_y; s) = -\sqrt{\frac{\beta_x}{\beta_0}}\, P_x x + \frac{1}{4}\frac{\beta_x'}{\beta_0} x^2 - \sqrt{\frac{\beta_y}{\beta_0}}\, P_y y + \frac{1}{4}\frac{\beta_y'}{\beta_0} y^2. \tag{2}$$

The new Hamiltonian becomes

$$H_2 = \frac{R}{2\beta_x}\left(\beta_0 p_x^2 + \frac{x^2}{\beta_0}\right) + \frac{R}{2\beta_y}\left(\beta_0 p_y^2 + \frac{y^2}{\beta_0}\right) + \frac{RB_y''}{6B\rho}\left[\left(\frac{\beta_x}{\beta_0}\right)^{3/2} x^3 - 3\left(\frac{\beta_x\beta_y^2}{\beta_0^3}\right)^{\frac{1}{2}} x y^2\right]. \tag{3}$$

In above, β_x and β_y are the horizontal and vertical beta-functions and β_0, the horizontal beta-function for some reference point, is introduced to ensure that x and y still carry the dimension of a length. In Eq. (3), the independent variable s has also been changed to the more convenient $\theta = s/R$ where R is the average radius of the storage ring.

This Hamiltonian is now solved exactly to zero order in sextupole strength by canonical transformation to the action-angle variables I_x, a_x and I_y, a_y. The generating function

$$G_2(a_x, p_x, a_y, p_y; \theta) = \sum_{z=x,y} \frac{1}{2}\beta_0 p_z^2 \cot\left[Q_z(\theta) + a_z\right] \tag{4}$$

is used so as to obtain

$$z = \sqrt{2 I_z \beta_0} \cos \left[Q_z(\theta) + a_z \right],$$

$$\beta_0 p_z = - \sqrt{2 I_z \beta_0} \sin \left[Q_z(\theta) + a_z \right], \tag{5}$$

with z = x or y (similarly in below), conforming with the convention of Collins[1]. In Ohnuma's paper[2] sine is used for z and cosine for $\beta_0 p_z$ instead. In above, $Q_z = \psi_z - \nu_z \theta$, where ν_z is the betatron tune and $\psi_z = \int^\theta (R/\beta_z) d\theta$ is the Floquet phase at location θ. We note that $\beta_0 p_z = dz/d\psi_z$ and is usually denoted by z'. After the transformation, the new Hamiltonian reads

$$H_3 = \nu_x I_x + \nu_y I_y + \text{sextupole term.} \tag{6}$$

The sextupole term, being periodic in θ, is now expanded as a Fourier series. With the help of Eq. (5), we get

$$\frac{RB_y''}{6B\rho} \left[\left(\frac{\beta_x}{\beta_0} \right)^{3/2} x^3 - 3 \left(\frac{\beta_x \beta_y^2}{\beta_0^3} \right)^{1/2} x y^2 \right]$$

$$= (2 I_x)^{3/2} \beta_0^{1/2} \sum_m \left(A_{3m} \sin q_{3m} + 3 A_{1m} \sin q_{1m} \right)$$

$$- (2 I_x)^{1/2} (2 I_y) \beta_0^{1/2} \sum_m \left(2 B_{1m} \sin p_{1m} + B_{+m} \sin p_{+m} + B_{-m} \sin p_{-m} \right), \tag{7}$$

with $q_{3m} = 3a_x - m\theta + \alpha_{3m}$, $q_m = a_x - m\theta + \alpha_{1m}$, $p_{1m} = a_x - m\theta + \beta_{1m}$, $p_{\pm m} = a_x \pm 2a_y - m\theta + \beta_{\pm m}$, and

$$
\begin{aligned}
A_{1m} e^{i\alpha_{1m}} &= \frac{i}{24\pi} \sum_k s_k e^{i(Q_x + m\theta)_k}, \\
A_{3m} e^{i\alpha_{3m}} &= \frac{i}{24\pi} \sum_k s_k e^{i(3Q_x + m\theta)_k}, \\
B_{1m} e^{i\beta_{1m}} &= \frac{i}{8\pi} \sum_k \bar{s}_k e^{i(Q_x + m\theta)_k}, \\
B_{\pm m} e^{i\beta_{\pm m}} &= \frac{i}{8\pi} \sum_k \bar{s}_k e^{i(Q_x \pm 2Q_y + m\theta)_k}.
\end{aligned}
\tag{8}
$$

The summations in Eq. (7) are over all integers m from $-\infty$ to $+\infty$. The summations in Eq. (8) are over all sextupoles at position θ_k along the ring. The sextupoles are assumed to have infinitesimal length ℓ_k with strengths

$$
s_k = \left(\frac{\beta_x^3}{\beta_0}\right)_k^{1/2} \frac{(B_y''\ell)_k}{2B\rho}, \qquad \bar{s}_k = \left(\frac{\beta_x \beta_y^2}{\beta_0}\right)_k^{1/2} \frac{(B_y''\ell)_k}{2B\rho}. \tag{9}
$$

The equations of motion are given by

$$
\begin{aligned}
\frac{dI_x}{d\theta} = -\frac{\partial H_3}{\partial a_x} = &-(2I_x)^{3/2} \beta_0^{1/2} \sum_m (3A_{3m} \cos q_{3m} + 3A_{1m} \cos q_{1m}) \\
&+ (2I_x)^{1/2} (2I_y) \beta_0^{1/2} \sum_m (2B_{1m} \cos p_{1m} + B_{+m} \cos p_{+m} + B_{-m} \cos p_{-m}),
\end{aligned}
\tag{10}
$$

$$
\frac{dI_y}{d\theta} = -\frac{\partial H_3}{\partial a_y} = (2I_x)^{\frac{1}{2}} (2I_y) \beta_0^{1/2} \sum_m (2B_{+m} \cos p_{+m} - 2B_{-m} \cos p_{-m}), \tag{11}
$$

$$\frac{da_x}{d\theta} = \frac{\partial H_3}{\partial I_x} = \nu_x + 3(2I_x)^{\frac{1}{2}}\beta_0^{\frac{1}{2}}\sum_m \left(A_{3m}\sin q_{3m} + 3A_{1m}\sin q_{1m}\right)$$
$$-(2I_x)^{-\frac{1}{2}}(2I_y)\beta_0^{1/2}\sum_m \left(2B_{1m}\sin p_{1m} + B_{+m}\sin p_{+m} + B_{-m}\sin p_{-m}\right), \tag{12}$$

$$\frac{da_y}{d\theta} = \frac{\partial H_3}{\partial I_y} = \nu_y - 2(2I_x)^{\frac{1}{2}}\beta_0^{\frac{1}{2}}\sum_m \left(2B_{1m}\sin p_{1m} + B_{+m}\sin p_{+m} + B_{-m}\sin p_{-m}\right). \tag{13}$$

The solution of Eqs. (12) and (13) gives, in the absence of sextupoles, $a_z = \nu_z\theta$ + constant. We choose

$$a_z = \nu_z\theta - \psi_z + \phi_z, \tag{14}$$

where ψ_z, designating the position of the particles along the ring, is the Floquet phase at position θ and ϕ_z is the instantaneous phase of the betatron oscillation. This becomes clear when substituted into Eq. (5) resulting in

$$z = \sqrt{2I_z\beta_0}\cos\phi_z \quad \text{and} \quad z' = -\sqrt{2I_z\beta_0}\sin\phi_z. \tag{15}$$

In Eq. (14), both ψ_z and ϕ_z are functions of θ but the difference ϕ_z-ψ_z is θ-independent.

Since we are interested in solutions accurate up to lowest order in s_k and $\bar{s}_k$ only, on the right hand sides of Eqs. (10) to (13), I_x and I_y can be considered as θ-independent and Eq. (14) can be substituted for a_z. Then all the four differential equations can be integrated easily. Denoting by δ the derivation from the situation when the sextupoles are absent, we obtain

$$\delta I_x = (2I_x)^{3/2} \beta_0^{1/2} \sum_m \left(\frac{3A_{3m}}{m-3\nu_x} \sin q_{3m} + \frac{3A_{1m}}{m-\nu_x} \sin q_{1m} \right)$$

$$- (2I_x)^{3/2} (2I_y) \beta_0^{1/2} \sum_m \left(\frac{2B_{1m}}{m-\nu_x} \sin p_{1m} + \frac{B_{+m}}{m-\nu_+} \sin p_{+m} + \frac{B_{-m}}{m-\nu_-} \sin p_{-m} \right), \qquad (16)$$

$$\delta I_y = -(2I_x)^{3/2} (2I_y) \beta_0^{1/2} \sum_m \left(\frac{2B_{+m}}{m-\nu_+} \sin p_{+m} - \frac{2B_{-m}}{m-\nu_-} \sin p_{-m} \right), \qquad (17)$$

$$\delta a_x = 3(2I_x)^{\frac{1}{2}} \beta_0^{\frac{1}{2}} \sum_m \left(\frac{A_{3m}}{m-3\nu_x} \cos q_{3m} + \frac{3A_{1m}}{m-\nu_x} \cos q_{1m} \right)$$

$$- (2I_x)^{-\frac{1}{2}} (2I_y) \beta_0^{\frac{1}{2}} \sum_m \left(\frac{2B_{1m}}{m-\nu_x} \cos p_{1m} + \frac{B_{+m}}{m-\nu_+} \cos p_{+m} + \frac{B_{-m}}{m-\nu_-} \cos p_{-m} \right), \qquad (18)$$

$$\delta a_y = -2(2I_x)^{\frac{1}{2}} \beta_0^{\frac{1}{2}} \sum_m \left(\frac{2B_{1m}}{m-\nu_x} \cos p_{1m} + \frac{B_{+m}}{m-\nu_+} \cos p_{+m} + \frac{B_{-m}}{m-\nu_-} \cos p_{-m} \right), \qquad (19)$$

where $\nu_\pm = \nu_x \pm 2\nu_y$. Exactly the same expressions can also be obtained by making a Moser transformation[4] from a_z, I_z to b_z, J_z so that J_z become constants of motion up to first order in s_k or $\bar{s}_k$. The required generating function

$$G_3(a_x, J_x, a_y, J_y; \theta)$$

$$= a_x J_x + a_y J_y - (2J_x)^{3/2} \beta_0^{1/2} \sum_m \left(\frac{A_{3m}}{m-3\nu_x} \cos q_{3m} + \frac{3A_{1m}}{m-\nu_x} \cos q_{1m} \right)$$

$$+ (2J_x)^{\frac{1}{2}} (2J_y) \beta_0^{\frac{1}{2}} \sum_m \left(\frac{2B_{1m}}{m-\nu_x} \cos p_{1m} + \frac{B_{+m}}{m-\nu_+} \cos p_{+m} + \frac{B_{-m}}{m-\nu_-} \cos p_{-m} \right)$$

can easily be obtained by solving the Hamilton-Jacobi equation. Equations (16) - (19) can then be derived by noting that

$$\delta I_z = I_z - J_z = \frac{\partial G_3}{\partial a_z} - J_z ,$$

$$\delta a_z = a_z - b_z = a_z - \frac{\partial G_3}{\partial J_z} .$$

Our final task is to simplify Eqs. (16) - (19) by doing the summation over m. This can be accomplished easily using the formula

$$\sum_{m=-\infty}^{\infty} \frac{e^{i(m\theta+b)}}{m-\nu} = \begin{cases} -\frac{\pi}{\sin\pi\nu} e^{i[b+\nu(\theta-\pi)]} & 0<\theta<2\pi , \\ -\pi\cot\pi\nu \; e^{ib} & \theta = 0 . \end{cases}$$

Take for example the terms involving A_{1m}. (The subscript x will be dropped for clarity whenever there is no ambiguity.) We have

$$\sum_m \frac{A_{1m} e^{iq_{1m}}}{m-\nu} = \frac{i}{24\pi} \sum_k s_k \sum_m \frac{e^{i(Q_k + m\theta_k - m\theta + a)}}{m-\nu}$$

$$= -\frac{i}{24\sin\pi\nu} \sum_k s_k \cdot \begin{cases} \exp i(a-\nu\theta+\psi_k-\pi\nu) & 0<\theta_k-\theta<2\pi \\ \exp i(a-\nu\theta+\psi_k+\pi\nu) & 0<\theta-\theta_k<2\pi \\ \cos\pi\nu \exp i(a-\nu\theta+\psi_k) & \theta=\theta_k \end{cases}$$

where use has been made of the relation $Q_k = \psi_k - \nu\theta_k$. Upon substituting $a = \nu\theta - \psi + \phi$ from Eq. (14), we get

$$\sum_m \frac{A_{1m} e^{iq_{1m}}}{m-\nu}$$

$$= -\frac{i}{24 \sin\pi\nu} \sum_k S_k \cdot \begin{cases} \exp i(\psi_k - \psi - \pi\nu + \phi) & 0 < \psi_k - \psi < 2\pi\nu \\ \exp i(\psi_k - \psi + \pi\nu + \phi) & 0 < \psi - \psi_k < 2\pi\nu \\ \cos\pi\nu \exp i\phi & \psi = \psi_k \end{cases}$$

$$= -\frac{i}{24 \sin\pi\nu} \sum_k S_k \cdot \begin{cases} [\cos(\psi_k - \psi - \pi\nu) + i\sin(\psi_k - \psi - \pi\nu)] e^{i\phi} \\ [\cos(\psi - \psi_k - \pi\nu) - i\sin(\psi - \psi_k - \pi\nu)] e^{i\phi} \\ \cos\pi\nu\, e^{i\phi} \end{cases}$$

$$= \frac{1}{3}\left[-iB_1(\psi_x) + A_1(\psi_x)\right] e^{i\phi_x}, \qquad (20)$$

where B_1 and A_1 are one set of distortion functions defined by Collins:

$$B_1(\psi_x) = \frac{1}{2\sin\pi\nu_x} \sum_k \frac{S_k}{4} \cos(|\psi_{xk} - \psi_x| - \pi\nu_x), \qquad 0 \le |\psi_{xk} - \psi_x| \le 2\pi\nu_x,$$

$$A_1(\psi_x) = B_1'(\psi_x), \qquad 0 < |\psi_{xk} - \psi_x| < 2\pi\nu_x.$$

They and others defined by Eq. (25) are in fact lattice functions due to the presence of sextupoles just as the β and α are lattice functions due to the presence of quadrupoles. They are periodic functions of the ring and closed after one revolution. Written as a vector, (B_1, A_1) rotates around the ring according to the angle equal to the phase advanced. At a sextupole of strength s_k, A_1 jumps by $s_k/4$ while B_1 remains continuous but exhibits a cusp.

In exactly the same way, we obtain

$$\sum_m \frac{A_{3m} e^{iq_{3m}}}{m-3\nu_x} = \frac{1}{3}(-iB_3 + A_3) e^{i\phi_x}, \tag{21}$$

$$\sum_m \frac{B_{1m} e^{ip_{1m}}}{m-\nu_x} = (-i\bar{B} + \bar{A}) e^{i\phi_x}, \tag{22}$$

$$\sum_m \frac{B_{+m} e^{ip_{+m}}}{m-\nu_+} = (-iB_S + A_S) e^{i\sigma}, \tag{23}$$

$$\sum_m \frac{B_{-m} e^{ip_{-m}}}{m-\nu_-} = (iB_D - A_D) e^{i\delta_0}, \tag{24}$$

where $\sigma = 2\phi_y+\phi_x$, $\delta_0 = 2\phi_y-\phi_x$ and the other four sets of distortion functions are defined by

$$B_3(\psi_x) = \frac{1}{2\sin 3\pi\nu_x} \sum_k \frac{s_k}{4} \cos 3(|\psi_{xk}-\psi_x| - \pi\nu_x), \quad 0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x$$

$$A_3(\psi_x) = B_3'(\psi_x),$$

$$\bar{B}(\psi_x) = \frac{1}{2\sin \pi\nu_x} \sum_k \frac{\bar{s}_k}{4} \cos(|\psi_{xk}-\psi_x| - \pi\nu_x), \quad 0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x$$

$$\bar{A}(\psi_x) = \bar{B}'(\psi_x),$$

$$B_{S,D}(2\psi_y \pm \psi_x) = \frac{1}{2\sin\pi(2\nu_y \pm \nu_x)} \sum_k \frac{\bar{S}_k}{4} \cos\Big[\,|(2\psi_{yk} \pm \psi_{xk}) - (2\psi_y \pm \psi_x)|$$

$$- \pi(2\nu_y \pm \nu_x)\Big] \qquad 0 \le |(2\psi_{yk} \pm \psi_{xk}) - (2\psi_y \pm \psi_x)| \le -2\pi\nu_\pm$$

$$A_{S,D}(2\psi_y \pm \psi_x) = B'_{S,D}(2\psi_y \pm \psi_x)\,. \tag{25}$$

From Eq. (5), we recall that the distortion of the amplitudes A_z and phases ϕ_z are given by $\delta A_z = \delta(2I_z\beta_0)^{\frac{1}{2}}$ and $\delta\phi_z = \delta a_z$. Using Eqs. (16) - (19) and (20) - (24), we arrive at

$$\delta A_x = -A_x^2\Big[(B_3\cos 3\phi_x - A_3\sin 3\phi_x) + (B_1\cos\phi_x - A_1\sin\phi_x)\Big]$$

$$+ A_y^2\Big[2(\bar{B}\cos\phi_x - \bar{A}\sin\phi_x) + (B_S\cos\sigma - A_S\sin\sigma) - (B_D\cos\delta_0 - A_D\sin\delta_0)\Big], \tag{26}$$

$$\delta\phi_x = A_x\Big[(B_3\sin 3\phi_x + A_3\cos 3\phi_x) + 3(B_1\sin\phi_x + A_1\cos\phi_x)\Big]$$

$$- \frac{A_y^2}{A_x}\Big[2(\bar{B}\sin\phi_x + \bar{A}\cos\phi_x) + (B_S\sin\sigma + A_S\cos\sigma) + (B_D\sin\delta_0 + A_D\cos\delta_0)\Big], \tag{27}$$

$$\delta A_y = 2A_xA_y\Big[(B_S\cos\sigma - A_S\sin\sigma) + (B_D\cos\delta_0 - A_D\sin\delta_0)\Big], \tag{28}$$

$$\delta\phi_y = -2A_x\Big[2(\bar{B}\sin\phi_x + \bar{A}\cos\phi_x) + (B_S\sin\sigma + A_S\cos\sigma) + (B_D\sin\delta_0 + A_D\cos\delta_0)\Big]. \tag{29}$$

The sextupoles will have horizontal bending effect on an off-axis particle. This will lead to a distortion of the ideal closed orbit. This can be obtained by separating out from Eq. (26) and (27),

$$\delta\mathcal{A}_x' = -2\mathcal{A}_x^2(B_1\cos\phi_x - A_1\sin\phi_x) + 2\mathcal{A}_y^2(\bar{B}\cos\phi_x - \bar{A}\sin\phi_x), \tag{30}$$

$$\mathcal{A}_x\delta\phi_x' = 2\mathcal{A}^2(B_1\sin\phi_x + A_1\cos\phi_x) - 2\mathcal{A}^2(\bar{B}\sin\phi_x + \bar{A}\cos\phi_x), \tag{31}$$

which correspond to a closed orbit distortion of

$$\delta x = 2(\mathcal{A}_y^2\bar{B} - \mathcal{A}_x^2 B_1), \tag{32}$$

$$\delta x' = 2(\mathcal{A}_y^2\bar{A} - \mathcal{A}_x^2 A_1), \tag{33}$$

where $x' = dx/d\psi_x$. Thus the distorted beam shape in phase space can be written as

$$x = \delta x + (\mathcal{A}_x + \delta\mathcal{A}_x)\cos(\phi_x + \delta\phi_x),$$

$$x' = \delta x' - (\mathcal{A}_x + \delta\mathcal{A}_x)\sin(\phi_x + \delta\phi_x),$$

$$y = (\mathcal{A}_y + \delta\mathcal{A}_y)\cos(\phi_y + \delta\phi_y),$$

$$y' = -(\mathcal{A}_y + \delta\mathcal{A}_y)\sin(\phi_y + \delta\phi_y),$$

where δA_y and $\delta\phi_y$ are given by Eqs. (28) and (29), δx and $\delta x'$ by Eqs. (32) and (33), δA_x and $\delta\phi_x$, by the differences of Eqs. (26), (27) and Eqs. (30), (31), or

$$\delta A_x = -A_x^2\left[(B_3\cos 3\phi_x - A_3\sin 3\phi_x) - (B_1\cos\phi_x - A_1\sin\phi_x)\right]$$
$$+ A_y^2\left[(B_s\cos\sigma - A_s\sin\sigma) - (B_D\cos\delta_0 - A_D\sin\delta_0)\right]$$

$$\delta\phi_x = A_x\left[(B_3\sin 3\phi_x + A_3\cos 3\phi_x) + (B_1\sin\phi_x + A_1\cos\phi_x)\right]$$
$$- \frac{A_y^2}{A_x}\left[(B_s\sin\sigma + A_s\cos\sigma) + (B_D\sin\delta_0 + A_D\cos\delta_0)\right].$$

These distortion formulas are exactly those given by Collins.

References

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